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# Finite $p$ -groups with two conjugacy length

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## 1 Introduction

Let  $G$  be a finite group. We denote by  $cd(G)$  and  $ccl(G)$  the sets of numbers which occur as the degrees of irreducible characters of  $G$  and as the lengths of conjugacy classes of  $G$  respectively. Isaacs and Passman prove in [2] that the commutator subgroup of any finite group  $G$  with  $cd(G) = \{1, m\}$  ( $m > 1$ ) is abelian. In this paper we will prove more strong analogous result for the set of conjugacy lengths. The class of group  $G$  with  $ccl(G) = \{1, m\}$  ( $m > 1$ ) has been introduced by Ito in [4] and he has shown that the study of such groups is reduced to that of  $p$ -groups for some prime  $p$ . Moreover, a result of Isaacs [1] follows that the central factor of any group of this class is of exponent  $p$ . Therefore any 2-group of this class is of nilpotent class 2. We show the following theorem:

**Main theorem** *Let  $G$  be a finite  $p$ -group such that  $ccl(G) = \{1, p^n\}$  ( $n \geq 1$ ). Then the commutator subgroup of  $G$  is an elementary abelian  $p$ -group.*

The result of Heineken in [5] follows that the conclusion of main theorem is equivalent that the nilpotent class of any group of this class is at most 3.

## 2 Main theorem

Let  $G$  be a finite group. We define  $[x, y] := x^{-1}y^{-1}xy$  and  $[x, y, z] := [[x, y], z]$  for all  $x, y, z \in G$ . The finite field of  $p$  elements will be denoted by  $F_p$ . The lower central series of  $G$  will be denoted by  $G_1 \leq G_2 \leq \dots$ , namely  $G_1 := G, G_2 := [G, G]$  and  $G_{i+1} := [G_i, G]$  ( $i \geq 2$ ). If  $c + 1$  is the least value of  $m$  satisfying  $G_m = 1$ , then  $c$  is called the nilpotent class of  $H$ . The nilpotency class of nilpotent group  $H$  will be denoted by  $c(H)$ .

The following result of Isaacs gives some useful corollaries.

**Theorem 2.1 (Isaacs [1])** *Let  $G$  be a finite group, which contains a proper normal subgroup  $N$  such that all of the conjugacy classes of  $G$  which lie outside of  $N$  have the same lengths. Then either  $G/N$  is cyclic, or else every nonidentity element of  $G/N$  has prime order.*

**Corollary 2.2** *Let  $G$  be a finite  $p$ -group such that  $\text{ccl}(G) = \{1, p^n\}$  ( $n \geq 1$ ). Then  $G/Z(G)$  is of exponent  $p$ .*

Note that  $G_{i-1}Z(G)/G_iZ(G)$  is an elementary abelian  $p$ -group for  $2 \leq i \leq c(G)$ .

**Corollary 2.3** *Let  $G$  be a finite 2-group such that  $\text{ccl}(G) = \{1, 2^n\}$  ( $n \geq 1$ ). Then  $G$  is of nilpotent class 2.*

**Corollary 2.4 (Verardi, Corollary 2.5[5])** *Let  $p$  be an odd prime. Let  $G$  be a finite  $p$ -group such that  $\text{ccl}(G) = \{1, p^n\}$  ( $n \geq 1$ ) and  $c(G) \geq 3$ . Then  $G_{c(G)-1}$  is an elementary abelian  $p$ -group.*

**Theorem 2.5 (Heineken, Theorem 2.7[5])** *Let  $G$  be a finite  $p$ -group such that  $\text{ccl}(G) = \{1, p^n\}$  ( $n \geq 1$ ). Then we have*

$$\{x \in G; xZ(G) \in Z(G/Z(G))\} = C_G(D(G)) = \{x \in G; C_G(x) \triangleleft G\}.$$

**Corollary 2.6** *Let  $G$  be a finite  $p$ -group of  $\text{ccl}(G) = \{1, p^n\}$  ( $n \geq 1$ ). Suppose that  $D(G)$  is abelian. Then  $G$  is of nilpotent class at most 3.*

*Proof.* Suppose that  $D(G)$  is abelian. By Theorem 2.5,  $D(G)Z(G)/Z(G) \leq C_G(D(G))Z(G)/Z(G) \leq Z(G/Z(G))$ . Therefore  $[D(G), G] \leq Z(G)$ . The proof is completed. ■

Let  $G$  be a finite  $p$ -group such that  $\text{ccl}(G) = \{1, p^n\}$  ( $n \geq 1$ ). Then corollaries 2.4 and 2.6 say that the commutator subgroup of  $G$  is abelian if and only if  $c(G) \leq 3$ .

*Proof of main theorem.* We will show that  $c(G) \leq 3$ . Suppose that  $p$  is an odd prime and  $c = c(G) \geq 3$ . Then there exist  $y \in G$  and  $z \in G_{c-2}$  such that  $[y, z] \in G_{c-1} - Z(G)$ . Now we put  $x_1 := [y, z]$ ,  $p^m := |G_{c-1}Z(G)/Z(G)|$  and  $G_{c-1}Z(G)/Z(G) = \langle x_1, x_2, \dots, x_m \rangle Z(G)/Z(G)$ . Note that  $G_{c-1}Z(G)/Z(G)$  is an elementary abelian  $p$ -group. We put a coset decomposition of  $G$  by  $C_G(x_1)$  as the following:

$$G = \bigcup_{\alpha_i=0}^{p-1} u_1^{\alpha_1} u_2^{\alpha_2} \cdots u_n^{\alpha_n} C_G(x_1)$$

and  $c_i := [x_1, u_i] \in Z(G)$  for  $1 \leq i \leq n$ . Since  $G_{c-1}$  is a elementary abelian  $p$ -group,  $|\langle c_1, \dots, c_n \rangle| = p^n$ . By Witt's identity, we have

$$[y, z, u_i][u_i, y, z][z, u_i, y] = 1.$$

Then  $[y, z, u_i] = c_i$  and we can write

$$[z, u_i] = x_1^{\beta_{i1}} x_2^{\beta_{i2}} \cdots x_m^{\beta_{im}} w \in G_{c-1}$$

for some  $\beta_{ij} \in F_p$  ( $1 \leq i \leq n, 1 \leq j \leq m$ ) and some  $w \in Z(G)$ . We put  $d_i := [x_i, y]$ , then

$$[z, u_i, y] = d_1^{\beta_{i1}} d_2^{\beta_{i2}} \dots d_m^{\beta_{im}}$$

for  $1 \leq i \leq n$ . Therefore we have

$$[u_i, y, z] = c_i^{-1} d_1^{-\beta_{i1}} d_2^{-\beta_{i2}} \dots d_m^{-\beta_{im}}$$

for  $1 \leq i \leq n$ .

We put  $p^s = |\langle c_i^{-1} d_1^{-\beta_{i1}} d_2^{-\beta_{i2}} \dots d_m^{-\beta_{im}}; 1 \leq i \leq n \rangle|$  and  $p^t = |\langle d_1, d_2, \dots, d_m \rangle|$ . Then, by  $\langle c_1, c_2, \dots, c_n \rangle \subset \langle d_1, d_2, \dots, d_m, c_i^{-1} d_1^{-\beta_{i1}} d_2^{-\beta_{i2}} \dots d_m^{-\beta_{im}}; 1 \leq i \leq n \rangle$ , we have  $s + t \geq n$ . We choose  $I = \{i_1, i_2, \dots, i_s\} \subset \{1, 2, \dots, n\}$  and  $J = \{j_1, j_2, \dots, j_t\} \subset \{1, 2, \dots, m\}$  such that  $|\langle c_i^{-1} d_1^{-\beta_{i1}} d_2^{-\beta_{i2}} \dots d_m^{-\beta_{im}}; i \in I \rangle| = p^s$  and  $|\langle d_{j_1}, \dots, d_{j_t} \rangle| = p^t$  respectively. Now we consider the subset

$$\bigcup_{\gamma_i=0}^{p-1} C_G(y) u_{i_1}^{\gamma_1} u_{i_2}^{\gamma_2} \dots u_{i_s}^{\gamma_s} x_{j_1}^{\gamma_{s+1}} x_{j_2}^{\gamma_{s+2}} \dots x_{j_t}^{\gamma_{s+t}}$$

of  $G$ . We claim that this sum is disjoint. We will show that if

$$[u_{i_1}^{\delta_1} \dots u_{i_s}^{\delta_s} x_{j_1}^{\delta_{s+1}} \dots x_{j_t}^{\delta_{s+t}}, y] = [u_{i_1}^{\varepsilon_1} \dots u_{i_s}^{\varepsilon_s} x_{j_1}^{\varepsilon_{s+1}} \dots x_{j_t}^{\varepsilon_{s+t}}, y],$$

then  $\delta_i = \varepsilon_i$  for  $1 \leq i \leq s + t$ . This equation is rewritten the following:

$$[u_{i_1}^{\delta_1} \dots u_{i_s}^{\delta_s}, y][x_{j_1}^{\delta_{s+1}} \dots x_{j_t}^{\delta_{s+t}}, y] = [u_{i_1}^{\varepsilon_1} \dots u_{i_s}^{\varepsilon_s}, y][x_{j_1}^{\varepsilon_{s+1}} \dots x_{j_t}^{\varepsilon_{s+t}}, y].$$

Then

$$\begin{aligned} [u_{i_1}^{\delta_1} \dots u_{i_s}^{\delta_s}, y][x_{j_1}^{\delta_{s+1}} \dots x_{j_t}^{\delta_{s+t}}, y] G_3 &= [u_{i_1}^{\varepsilon_1} \dots u_{i_s}^{\varepsilon_s}, y][x_{j_1}^{\varepsilon_{s+1}} \dots x_{j_t}^{\varepsilon_{s+t}}, y] G_3 \\ [u_{i_1}^{\delta_1} \dots u_{i_s}^{\delta_s}, y] G_3 &= [u_{i_1}^{\varepsilon_1} \dots u_{i_s}^{\varepsilon_s}, y] G_3 \\ [u_{i_1}^{\delta_1}, y] \dots [u_{i_s}^{\delta_s}, y] G_3 &= [u_{i_1}^{\varepsilon_1}, y] \dots [u_{i_s}^{\varepsilon_s}, y] G_3 \\ G_3 &= [u_{i_1}, y]^{\varepsilon_1 - \delta_1} \dots [u_{i_s}, y]^{\varepsilon_s - \delta_s} G_3. \end{aligned}$$

By  $[G_3, z] = 1$ , we have

$$\begin{aligned} 1 &= [[u_{i_1}, y]^{\varepsilon_1 - \delta_1} \dots [u_{i_s}, y]^{\varepsilon_s - \delta_s}, z] \\ &= \prod_{k=1}^s (c_{i_k}^{-1} d_1^{-\beta_{i_k 1}} d_2^{-\beta_{i_k 2}} \dots d_m^{-\beta_{i_k m}})^{\varepsilon_k - \delta_k}. \end{aligned}$$

By the choice of  $I$ , we have  $\delta_i = \varepsilon_i$  for  $1 \leq i \leq s$ . So it is enough to verify

$$[x_{j_1}^{\delta_{s+1}} \dots x_{j_t}^{\delta_{s+t}}, y] = [x_{j_1}^{\varepsilon_{s+1}} \dots x_{j_t}^{\varepsilon_{s+t}}, y].$$

$$\begin{aligned}
[x_{j_1}^{\delta_{s+1}} \cdots x_{j_t}^{\delta_{s+t}}, y] &= [x_{j_1}^{\varepsilon_{s+1}} \cdots x_{j_t}^{\varepsilon_{s+t}}, y] \\
[x_{j_1}, y]^{\delta_{s+1}} \cdots [x_{j_t}, y]^{\delta_{s+t}} &= [x_{j_1}, y]^{\varepsilon_{s+1}} \cdots [x_{j_t}, y]^{\varepsilon_{s+t}} \\
d_{j_1}^{\delta_{s+1}} \cdots d_{j_t}^{\delta_{s+t}} &= d_{j_1}^{\varepsilon_{s+1}} \cdots d_{j_t}^{\varepsilon_{s+t}} \\
d_{j_1}^{\delta_{s+1}-\varepsilon_{s+1}} \cdots d_{j_t}^{\delta_{s+t}-\varepsilon_{s+t}} &= 1.
\end{aligned}$$

Hence, by the choice of  $J$ , we have  $\delta_i = \varepsilon_i$  for  $s+1 \leq i \leq s+t$ . Therefore our claim now follows. Then, by  $|G : C_G(y)| = p^n$  and  $s+t \geq n$ , we have  $s+t = n$ . Therefore

$$\bigcup_{\gamma_i=0}^{p-1} C_G(y) u_{i_1}^{\gamma_1} u_{i_2}^{\gamma_2} \cdots u_{i_s}^{\gamma_s} x_{j_1}^{\gamma_{s+1}} x_{j_2}^{\gamma_{s+2}} \cdots x_{j_t}^{\gamma_{s+t}}$$

is a right coset decomposition of  $G$  by  $C_G(y)$ . Then we consider the following:

$$[u_{i_1}^{\gamma_1} u_{i_2}^{\gamma_2} \cdots u_{i_s}^{\gamma_s} x_{j_1}^{\gamma_{s+1}} x_{j_2}^{\gamma_{s+2}} \cdots x_{j_t}^{\gamma_{s+t}}, y]$$

for  $\gamma_i \in F_p$  ( $1 \leq i \leq s+t$ ). By the choice of  $I$ , if  $(\gamma_1, \dots, \gamma_s) \neq (0, \dots, 0)$ , then this is contained in  $G_2 - G_3$ , or else in  $Z(G)$ . Hence  $\{[g, y]; g \in G\} \subset (G_2 - G_3) \cup Z(G)$ . Therefore, by  $[y, z] = x_1 \in G_{c-1} - Z(G)$ , we have  $c = 3$ . By Theorem 2.4, the proof is now completed. ■

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